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Traveling wave solutions for some factorized nonlinear PDEs

Octavio Cornejo-Pérez

Facultad de Ingeniería, Universidad Autónoma de Querétaro, Centro Universitario Cerro de las Campanas, 76010 Santiago de Querétaro, Mexico

E-mail: octavio.cornejo@uaq.mx

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Abstract

In this work, some new special traveling wave solutions of the convective Fisher equation, the time-delayed Burgers–Fisher equation, the Burgers–Fisher equation and a nonlinear dispersive–dissipative equation (Kakutani and Kawahara 1970 *J. Phys. Soc. Japan* **29** 1068) are obtained through the factorization technique. All of them share the same type of factorization scheme, which reduces the original equation to a Riccati equation of the same kind, whose general solution is given in terms of Bessel and Neumann functions. In addition, some novel particular solutions of the nonlinear dispersive–dissipative equation are provided.

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1. Introduction

The search for exact solutions of nonlinear differential equations is an active field of research because they describe many different processes in several branches of science such as physics, biology and chemistry. Many methods have been developed to find analytical solutions of nonlinear ordinary differential equations (ODEs) and nonlinear partial differential equations (PDEs). Some of them are the truncation procedure in Painlevé analysis [1], the Hirota bilinear method [2], the tanh function method [3, 4], the Jacobi elliptic function method [5] and the Prelle–Singer method [6, 7]. The factorization method is a well-known technique used to find exact solutions of linear second-order ODEs in an algebraic manner [8]. In recent times, the factorization technique has been applied to find exact solutions of nonlinear ODEs and nonlinear PDEs in the context of traveling waves [9–14].

In this paper, some special traveling wave solutions of the following important nonlinear PDEs are obtained through the extended factorization technique [13]: the convective Fisher equation, the time-delayed Burgers–Fisher equation, the Burgers–Fisher equation and the nonlinear dispersive–dissipative equation as studied by Kakutani and Kawahara [15]. All of them are expressed as ODEs if the transformation to the traveling variable $z = x - vt$ is performed, and they share the same type of factorization scheme. This scheme is briefly described in the following.

Let us consider the nonlinear second-order ODE

$$\ddot{u} + g(u)\dot{u} + F(u) = 0, \tag{1}$$

where the dot means the derivative $D = \frac{\mathrm{d}}{\mathrm{d}z}$ and $g(u)$ and $F(u)$ are arbitrary functions of u. Equation (1) can be factorized [10, 11] in the following way:

$$[D - f_2(u)][D - f_1(u)]u = 0, \tag{2}$$

under the conditions

$$f_1 + f_2 + \frac{\mathrm{d}f_1}{\mathrm{d}u}u = -g(u), \tag{3}$$

$$f_1 f_2 u = F(u). \tag{4}$$

Then, a particular solution of the factorized equation (2) is obtained through the compatible first-order ODE

$$\dot{u} - f_1(u)u = 0. \tag{5}$$

Let us consider now the scheme proposed by Wang and Li in [13] for the extended factorization technique. Assuming $[D - f_1(u)]u = \Omega(z)$ yields the following coupled ODEs for equation (2):

$$\dot{\Omega} - f_2(u)\Omega = 0, \tag{6}$$

$$\dot{u} - f_1(u)u = \Omega(z), \tag{7}$$

which can be rewritten as

$$\dot{u} = f_1(u)u + \exp\left(\int f_2(u)\,\mathrm{d}z\right). \tag{8}$$

If one is able to solve equation (8) and obtain all its solutions, then all single traveling wave solutions of equation (2) will be derived.

Let the factorizing function f_2 be a constant, $f_2 = a_2 \equiv$ const., then the following coupled ODEs are obtained:

$$\dot{\Omega} - a_2\Omega = 0, \tag{9}$$

$$\dot{u} = a_2^{-1}F(u) + \Omega(z). \tag{10}$$

Equation (9) is a homogenous linear first-order ODE with the solution

$$\Omega(z) = c_1\,\mathrm{e}^{a_2 z}, \tag{11}$$

where c_1 is an integration constant. Therefore, the system (9) and (10) can be rewritten in the form

$$\dot{u} = a_2^{-1}F(u) + c_1\,\mathrm{e}^{a_2 z}, \tag{12}$$

whose general solution is also the solution of the factorized equation (2) when $f_2 = a_2$.

2. The convective Fisher equation

The convective Fisher equation studied by Schönborn *et al* [16] is given as

$$u_t = \tfrac{1}{2}u_{xx} - \mu u u_x + u(1-u). \tag{13}$$

The positive parameter μ serves to tune the relative strength of convection. As pointed by Reyes and Rosu [17], the convective Fisher equation was used by Walsh *et al* [18] in 1995 to simulate the population mobility according to spatial gradients in the food supply. Other applications could be in flame propagation when convection is taken into account and in branching Brownian motion under some spatial bias gradients.

If the transformation to the traveling variable $z = x - vt$ in equation (13) is performed, then we get

$$\ddot{u} + 2(v - \mu u)\dot{u} + 2u(1-u) = 0. \tag{14}$$

It can be easily shown that if the factorizing functions $f_1(u) = -\mu(1-u)$ and $f_2 = -2/\mu$ are chosen, then equation (14) admits the factorization

$$\left[D + \frac{2}{\mu}\right][D + \mu(1-u)]u = 0. \tag{15}$$

The velocity of the traveling wave is obtained by comparing both sides of equation (3) once the factorizing functions f_1 and f_2 are defined, giving as result $v = \mu/2 + \mu^{-1}$ [10]. The substitution of f_1 and f_2 into equation (12) leads to the following Riccati equation:

$$\dot{u} = \mu u^2 - \mu u + c_1\,\mathrm{e}^{-2z/\mu}, \tag{16}$$

whose general solution is given in terms of Bessel J_n and Neumann N_n functions,

$$u = -\sqrt{\frac{c_1}{\mu}}\,\mathrm{e}^{-z/\mu}\frac{J_{1-\frac{\mu^2}{2}}(\xi(z)) + c_2 N_{1-\frac{\mu^2}{2}}(\xi(z))}{J_{-\frac{\mu^2}{2}}(\xi(z)) + c_2 N_{-\frac{\mu^2}{2}}(\xi(z))}, \tag{17}$$

where $\xi(z) = \sqrt{c_1}\mu^{3/2}\,\mathrm{e}^{-z/\mu}$ and c_2 is an integration constant. The exact solution (17) is also a special solution of the convective Fisher equation (13).

The Neumann N_n function is singular if the argument is zero. Therefore, we can set $c_2 = 0$ in equation (17) and consider only the quotient of Bessel J_n functions to obtain

$$u = -\sqrt{\frac{c_1}{\mu}}\,\mathrm{e}^{-z/\mu}\frac{J_{1-\frac{\mu^2}{2}}(\xi(z))}{J_{-\frac{\mu^2}{2}}(\xi(z))}. \tag{18}$$

Obviously, the order of the Bessel functions will depend on the specific value of the parameter μ. They can be represented in the following series expressions [19]:

$$J_n(\xi) = \sum_{s=0}^{\infty}\frac{(-1)^s}{s!(n+s)!}\left(\frac{\xi}{2}\right)^{n+2s} \tag{19}$$

and

$$J_{-n}(\xi) = \sum_{s=0}^{\infty}\frac{(-1)^s}{s!(s-n)!}\left(\frac{\xi}{2}\right)^{2s-n}, \tag{20}$$

for integer n. Equations (19) and (20) may be used with n replaced by ν to define $J_\nu(\xi)$ and $J_{-\nu}(\xi)$ for non-integer ν.

The asymptotic limit of the Bessel function $J_n(\xi(z))$ is given as $\lim_{z\to\infty} J_n(\xi(z)) = \lim_{\xi\to 0} J_n(\xi)$. For small arguments ξ, one obtains $J_n \to \frac{1}{\Gamma(n+1)}\left(\frac{\xi}{2}\right)^n$, for $n > 0$. A similar

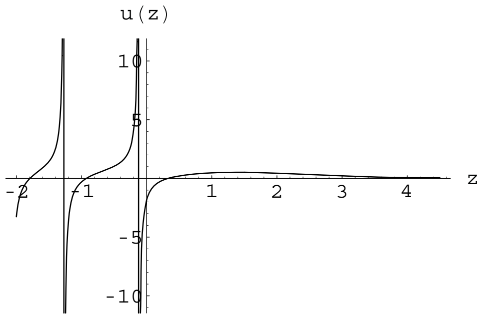

Figure 1. Plot of solution (17) for $\mu = 3.5$, $c_1 = 2$ and $c_2 = 0$, within the interval $z\epsilon(-2, 4.5)$. The solution goes to zero as $z \to \infty$.

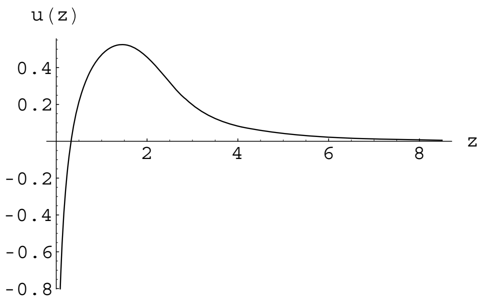

Figure 2. Plot of solution (17) within $z\epsilon(0, 8.5)$. Parameter values as in figure 1.

approach starting from equation (20) for the case J_{-n} could be obtained. Therefore, the asymptotic behavior of solution (18) as $z \to \infty$ will be given by

$$u(\infty) = \lim_{z\to\infty} -\sqrt{\frac{c_1}{\mu}}\, \mathrm{e}^{-z/\mu} \frac{J_{1-\frac{\mu^2}{2}}\left(\sqrt{c_1}\mu^{3/2}\,\mathrm{e}^{-z/\mu}\right)}{J_{-\frac{\mu^2}{2}}\left(\sqrt{c_1}\mu^{3/2}\,\mathrm{e}^{-z/\mu}\right)}. \tag{21}$$

Plots of solution (17) are given in figures 1 and 2. As we can see, the solution breaks down and blows up for some finite values of z. This class of blow-up/break-down solutions have been found and related to the breaking wave phenomenon in nonlinear wave equations [20–23].

Some particular solutions of the convective Fisher equation can be obtained from the Riccati equation (16). If we set $c_1 = 0$, then we get

$$\dot{u} = \mu u^2 - \mu u, \tag{22}$$

whose integration provides the following Reyes–Rosu (RR) solution [17]:

$$u_\lambda = [1 \pm \mathrm{e}^{\mu(z-z_0)}]^{-1} + \frac{\mathrm{e}^{-\mu(z-z_0)}}{[\mathrm{e}^{-\mu(z-z_0)} \pm 1][\lambda(\mathrm{e}^{-\mu(z-z_0)} \pm 1) - 1]}, \tag{23}$$

where z_0 is an integration constant and u_λ represents a one-parameter family of kink-type particular solutions of equation (13).

3. The time-delayed Burgers–Fisher equation

The Burgers–Fisher equation [14, 24, 25]

$$u_t = u_{xx} - puu_x + qu(1-u) \tag{24}$$

describes the interaction between reaction mechanisms, convection effects and diffusion transports. This description is modified if memory effects in diffusive processes are taken into account.

The time-delayed Burgers–Fisher equation [14, 25], which is a reaction–diffusion–convection equation with finite memory transport, is given by

$$\tau u_{tt} + \left(1 - \tau \frac{\mathrm{d}f}{\mathrm{d}u}\right) u_t = u_{xx} - puu_x + f(u), \qquad f(u) = qu(1-u), \tag{25}$$

where p and q are real constant parameters, and τ represents the delay time. The time-delayed Burgers–Fisher equation is important in the context of chemical kinetics, mathematical biology and turbulence.

The transformation to the traveling variable $z = x - vt$ in equation (25) yields

$$\ddot{u} + \frac{1}{1-v^2\tau}[v(1-\tau q) + (2v\tau q - p)u]\dot{u} + \frac{q}{1-v^2\tau} u(1-u) = 0. \tag{26}$$

If we choose the factorizing functions $f_1 = a_1 \frac{q}{1-v^2\tau}(1-u)$ and $f_2 = a_1^{-1}$, where a_1 is an arbitrary constant, then we get $a_1 = v\tau - \frac{p}{2q}$ and the velocity of the traveling wave $v = \frac{p^2+4q}{2p(1+q\tau)}$ from equation (3). Equation (26) admits the factorization

$$\left(D - \frac{2q}{2v\tau q - p}\right)\left(D - (v\tau q - p/2)\frac{1-u}{1-v^2\tau}\right) u = 0. \tag{27}$$

According to equation (12) we obtain the following Riccati equation:

$$\dot{u} = -\beta u^2 + \beta u + c_1 \,\mathrm{e}^{z/a_1}, \tag{28}$$

where $\beta = \frac{2v\tau q - p}{2(1-v^2\tau)}$, with the general solution given as

$$\begin{aligned} u &= -\sqrt{\frac{-c_1}{\beta}}\,\mathrm{e}^{z/2a_1} \frac{J_{1+\beta a_1}(\xi(z)) + c_2 N_{1+\beta a_1}(\xi(z))}{J_{\beta a_1}(\xi(z)) + c_2 N_{\beta a_1}(\xi(z))} + 1, \\ \xi(z) &= 2a_1\sqrt{-c_1\beta}\,\mathrm{e}^{z/2a_1}, \end{aligned} \tag{29}$$

where c_2 is an integration constant.

Once again we take $c_2 = 0$ to discard the Neumann N_n functions. The asymptotic analysis of this solution provides $u \to 1$ as $z \to -\infty$.

Plots of solution (29), given in figures 3 and 4, show breakdown and blowup for some finite values of z.

Some particular solutions of the time-delayed Burgers–Fisher equation can also be obtained from the Riccati equation (28). If we set $c_1 = 0$, then we get

$$\dot{u} = -\beta u^2 + \beta u. \tag{30}$$

Integration of equation (30) provides the following RR solution [14] of equation (25):

$$u_\lambda = [1 + \mathrm{e}^{-\beta(z-z_0)}]^{-1} + \frac{\mathrm{e}^{\beta(z-z_0)}}{[1+\mathrm{e}^{\beta(z-z_0)}][\lambda(1+\mathrm{e}^{\beta(z-z_0)}) - 1]}, \tag{31}$$

where z_0 is an integration constant.

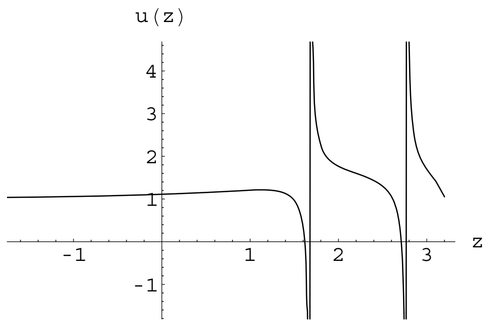

Figure 3. Plot of solution (29) for $\tau = 2$, $p = 1$, $q = 5$, $c_1 = 1$ and $c_2 = 0$, within the interval $z\epsilon(-1.8, 3.2)$. The solution tends to 1 as $z \to -\infty$.

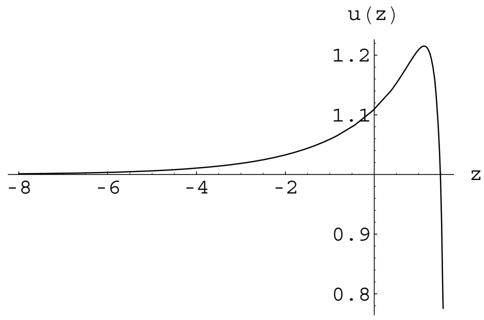

Figure 4. Plot of solution (29) within $z\epsilon(-8, 1.5)$. Parameter values as in figure 3.

3.1. The Burgers–Fisher equation

Equation (25) can be reduced to the Burgers–Fisher equation by setting $\tau = 0$,

$$u_t = u_{xx} - puu_x + qu(1-u). \tag{32}$$

The transformation to the traveling variable $z = x - vt$ yields the equation

$$\ddot{u} + (v - pu)\dot{u} + qu(1-u) = 0, \tag{33}$$

which admits the factorization

$$\left(D + \frac{2q}{p}\right)\left(D + \frac{p}{2}(1-u)\right)u = 0. \tag{34}$$

The factorizing functions are $f_1 = -\frac{p}{2}(1-u)$ and $f_2 = -\frac{2q}{p}$, and the velocity of the traveling wave is $v = \frac{p^2+4q}{2p}$. The substitution of f_1 and f_2 into equation (12) leads to the Riccati equation

$$\dot{u} = \frac{p}{2}u^2 - \frac{p}{2}u + c_1\,\mathrm{e}^{-\frac{2q}{p}z}, \tag{35}$$

with the general solution

$$u = -\sqrt{\frac{2c_1}{p}}\,\mathrm{e}^{-\frac{q}{p}z}\frac{J_{1-\frac{p^2}{4q}}(\xi(z)) + c_2N_{1-\frac{p^2}{4q}}(\xi(z))}{J_{-\frac{p^2}{4q}}(\xi(z)) + c_2N_{-\frac{p^2}{4q}}(\xi(z))}, \qquad \xi(z) = \frac{\sqrt{2c_1}}{2q}p^{3/2}\,\mathrm{e}^{-\frac{q}{p}z}, \tag{36}$$

and c_2 is an integration constant.

The RR solution of equation (32) is given by

$$u_\lambda = [1 + \mathrm{e}^{p(z-z_0)/2}]^{-1} + \frac{\mathrm{e}^{-p(z-z_0)/2}}{[1 + \mathrm{e}^{-p(z-z_0)/2}][\lambda(1 + \mathrm{e}^{-p(z-z_0)/2}) - 1]}, \tag{37}$$

where z_0 is an integration constant.

4. The nonlinear dispersive–dissipative equation

In this section, by the same procedure, several traveling wave solutions are obtained for the dispersive–dissipative equation as studied by Kakutani and Kawahara [15]. It has been derived by analyzing a two-fluid plasma model consisting of cold ions and warm electrons and describes weak nonlinear ion-acoustic waves. The equation is given as follows

$$u_t + uu_x + bu_{xxx} - a(u_t + muu_x)_x = 0, \tag{38}$$

where a, b and m are real constant parameters; or in the traveling frame

$$\dddot{u} + a(v - mu)\ddot{u} - am\dot{u}^2 + (u - v)\dot{u} = 0. \tag{39}$$

Equation (39) is integrated once to obtain

$$b\ddot{u} + a(v - mu)\dot{u} + \tfrac{1}{2}u^2 - vu + k = 0, \tag{40}$$

where k is an integration constant. Equation (40) can be factorized in the form (2) only if $k = 0$, which is a very restrictive condition. To avoid this constraint we apply the transformation $w(z) = u(z) + \delta$, where δ is a constant to be determined. Equation (40) can now be rewritten as

$$\ddot{w} + \frac{a}{b}[(v + m\delta) - mw]\dot{w} + \frac{1}{2b}[w^2 - 2(v + \delta)w] = 0, \tag{41}$$

where

$$\delta = -v \pm \sqrt{v^2 - 2k}, \tag{42}$$

and $k \leqslant v^2/2$ restricts δ to be a real constant. The factorizing functions are chosen as $f_1 = \frac{a_1}{2b}[w - 2(v + \delta)]$ and $f_2 = a_1^{-1}$. Hence, equation (3) provides $a_1 = am$, and the velocity of the traveling wave $v = \frac{b}{(am)^2 - a^2m}$. Equation (41) admits the factorization

$$\left(D - \frac{1}{am}\right)\left(D - \frac{am}{2b}[w - 2(v + \delta)]\right) w = 0. \tag{43}$$

The substitution of the factorizing functions into equation (12) gives the following Riccati equation,

$$\dot{w} = \frac{am}{2b}w^2 - \frac{am}{b}(v + \delta)w + c_1\,\mathrm{e}^{z/am}, \tag{44}$$

or

$$\dot{w} = \frac{am}{2b}w^2 \pm \frac{am}{b}\sqrt{v^2 - 2k}w + c_1\,\mathrm{e}^{z/am}, \tag{45}$$

whose general solution is given as

$$\begin{aligned} w_\pm &= \frac{b}{(am)^2}\xi(z)\frac{J_{1\mp\eta}(\xi(z)) + c_2N_{1\mp\eta}(\xi(z))}{J_{\mp\eta}(\xi(z)) + c_2N_{\mp\eta}(\xi(z))}, \\ \xi(z) &= \sqrt{\frac{2c_1}{b}}(am)^{3/2}\,\mathrm{e}^{z/2am}, \\ \eta &= \frac{b\sqrt{v^2 - 2k}}{(am)^2}, \end{aligned} \tag{46}$$

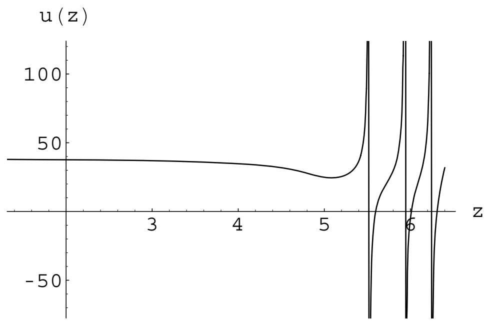

Figure 5. Plot of solution (47) u_+ for $a = 1, b = 1, m = 1.05, k = 0, c_1 = 1$ and $c_2 = 0$, within the interval $z\epsilon(-2, 6.4)$. The solution goes to a constant value $2v$ as $z \to -\infty$.

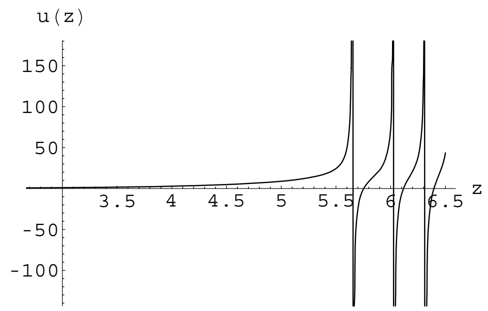

Figure 6. Plot of solution (47) u_- for $a = 1, b = 1, m = 1.05, k = 0, c_1 = 1$ and $c_2 = 0$, within the interval $z\epsilon(0, 6.5)$. The solution goes to zero as $z \to -\infty$.

and c_2 is an integration constant. Returning to the original function $u(z) = w(z) - \delta$ by combining equations (46) and (42), we get the following special solution of equation (38):

$$u_\pm = \frac{b}{(am)^2}\xi(z)\frac{J_{1\mp\eta}(\xi(z)) + c_2N_{1\mp\eta}(\xi(z))}{J_{\mp\eta}(\xi(z)) + c_2N_{\mp\eta}(\xi(z))} + v \pm \sqrt{v^2 - 2k}. \tag{47}$$

The solution (47) is a more general result than that obtained through other means by Isidore in [26]. It is also a different result from those obtained by Wang and Li [13]. They obtained the parametric form solution of equation (38) by means of an Abel equation of the second kind and one particular solution from the compatible first-order ODE (5).

Due to the fact that the Neumann N_n function is singular if the argument is zero we can take $c_2 = 0$ in equation (47). The asymptotic behavior for this solution is $u \to v \pm \sqrt{v^2 - 2k}$ as $z \to -\infty$.

Plots of equation (47) are displayed in figures 5 and 6. These solutions show breakdown and blowup for some finite values of z.

Several special cases of equation (45) provide particular solutions of equation (38):

(1) If we set $k = v^2/2$, then we obtain the following particular solution:

$$u = \frac{b}{(am)^2}\xi(z)\frac{J_1(\xi(z)) + c_2N_1(\xi(z))}{J_0(\xi(z)) + c_2N_0(\xi(z))} + v. \tag{48}$$

(2) If we choose $c_1 = 0$, then the integration of the resulting Riccati equation provides the following set of RR solutions:

$$u_{\lambda_{1,2}} = u_1 + \frac{2r}{-1 \pm \mathrm{e}^{-\frac{amr}{b}(z-z_0)}} + v + \sqrt{v^2 - 2k} \tag{49}$$

and

$$u_{\lambda_{3,4}} = u_1 + \frac{2r}{1 \pm \mathrm{e}^{\frac{amr}{b}(z-z_0)}} + v - \sqrt{v^2 - 2k}, \tag{50}$$

where

$$u_1 = \frac{2r\,\mathrm{e}^{\frac{amr}{b}(z-z_0)}}{2r\lambda(\mp 1 + \mathrm{e}^{\frac{amr}{b}(z-z_0)})^2 + (\mp 1 + \mathrm{e}^{\frac{amr}{b}(z-z_0)})},$$

and z_0 is an integration constant.

(3) Choosing $c_1 = 0$ and $k = v^2/2$ leads to the following RR solution:

$$u_\lambda = -\frac{2b}{am(z-z_0)} + \frac{2b}{2b\lambda(z-z_0)^2 + am(z-z_0)}. \tag{51}$$

5. Conclusion

This work presents a series of novel special traveling wave solutions of the following nonlinear PDEs: the convective Fisher equation, the time-delayed Burgers–Fisher equation, the Burgers–Fisher equation and a dispersive–dissipative equation [15]. These solutions have been found through the extended factorization method proposed by Wang and Li [13]. In addition, some novel particular solutions of the dispersive–dissipative equation are provided. The type of factorized equations studied here shares the same kind of factorizing function $f_2 \equiv$ const. If the factorizing function f_1 is a linear function of the dependent variable u, then equation (12) turns out to be a Riccati equation whose general solution is in terms of Bessel and Neumann functions. However, it is worth mentioning that if the factorizing function f_1 is a quadratic function of u, then equation (12) will become an Abel equation of the first kind. The special traveling wave solutions obtained here by the extended factorization technique show a wave breaking phenomenon [20–23] that deserves future work. The factorization scheme used to find the special solutions of the PDEs can also be applied to the study of other important nonlinear equations, for instance, the generalized Burgers–Fisher equation, the time-delayed convective Fisher equation [14], and in the context of ODEs, the Duffing—van der Pol oscillator equation [27], and the equation for the Hubble function H which arises in the study of causal bulk viscous cosmological models [28].

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